arXiv:cond-mat/0207009v1 [cond-mat.supr-con] 29 Jun 2002

# On the stability of the critical state in composite superconductors with an inhomogeneous temperature profile

**N.A. Taylanov**

*Theoretical Physics Department,*
*Scientific Research Institute for Applied Physics,*
*National University of Uzbekistan,*
*Tashkent, 700174, Uzbekistan*

*E-mail: taylanov@iaph.tkt.uz*

## Abstract

The problem of the thermal and magnetic destruction of the critical state in composite superconductors is investigated. The initial distributions of temperature and electromagnetic field are assumed to be essentially inhomogeneous. The limit of the thermomagnetic instability in quasi-stationary approximation is determined. The obtained integral criterion, unlike the analogous criterion for a homogeneous temperature profile, is shown to take into account the influence of any part of the superconductor on the threshold for critical-state instability.

**Key words**: Critical state, flux flow, flux jump, instability.

While dealing with instabilities of the critical state in hard superconductors, the character of the temperature distribution $T(x,t)$ and that of the electromagnetic field $\vec{E}(x,t)$ are of substantial practical interest [1]. This derives from the fact that thermal and magnetic distractions of the critical state caused by Joule self-heating are defined by the initial temperature and electromagnetic field distributions. Hence, the form of the temperature profile may noticeably influence the criteria of critical-state stability with respect to jumps in the magnetic flux in a superconductor. Earlier [2], in dealing with this problem, it was usually assumed that the spatial distribution of temperature and field were either homogeneous or slightly inhomogeneous. However, in reality, physical parameters of superconductors may be inhomogeneous along the sample as well as in its cross-sectional plane. Such inhomogeneities can appear due to different physical reasons. First, the vortex structure pinning can be inhomogeneous due to the existence of weak bonds in the superconductor. Second, inhomogenety of the properties may be caused by their dependence on the magnetic field $H$. Indeed, the field $H$ influences many physical quantities, such as the critical current density $j_c$, the differential conductivity $\sigma_d$, and the heat conductivity $k$.

In the present paper, the temperature distribution in the critical state of composite superconductors is investigated in the quasi-stationary approximation. It is shown that the temperature profile can be essentially inhomogeneous, which affects the conditions of initiation of magnetic flux jumps.

The evolution of thermal ($T$) and electromagnetic ($\vec{E}, \vec{H}$) perturbations in superconductors is described by a nonlinear heat conduction equation [3],

$$\nu \frac{dT}{dt} = \nabla[\kappa \nabla T] + \vec{j}\vec{E}, \tag{1}$$

a system of Maxwell equations,

$$rot\vec{E} = -\frac{1}{c}\frac{d\vec{H}}{dt}, \tag{2}$$

$$rot\vec{H} = \frac{4\pi}{c}\vec{j} \tag{3}$$

and a critical state equation

$$\vec{j} = \vec{j}_c(T, \vec{H}) + \vec{j}_r(\vec{E}). \tag{4}$$

Here $\nu = \nu(T)$ is the specific heat, $\kappa = \kappa(T)$ is the thermal conductivity respectively; $\vec{j}_c$ is the critical current density and $\vec{j}_r$ is the active current density.

We use the Bean-London critical state model to describe the $j_c(T, H)$ dependence, according to which $j_c(T) = j_0[1 - a(T - T_0)]$ [4], where the parameter $a$ characterizes thermally activated weakening of Abrikosov vortex pinning on crystal lattice defects, $j_0$ is the equilibrium current density, and $T_0$ is the temperature of the superconductor.

The dependence $j_r(E)$ in the region of sufficiently strong electric fields ($E \geq E_f$; where $E_f$ is the limit of the linear region of the current-voltage characteristic of the sample [2]) can be approximated by a piecewise-linear function $j_r \approx \sigma_f E$, where $\sigma_f = \frac{\eta c^2}{H\Phi_0} \approx \frac{\sigma_n H_{c_2}}{H}$ is the effective conductivity in the flux flow regime and $\eta$ is the viscous coefficient, $\Phi_0 = \frac{\pi hc}{2e}$ is the magnetic flux quantum, $\sigma_n$ is the conductivity in the normal state, $H_{c_2}$ is the upper critical magnetic field. In the region of the weak fields ($E \leq E_f$), the function $j_r(E)$ is nonlinear. This nonlinearity is associated with thermally activated creep of the magnetic flux [5].

Let us consider a superconducting sample placed into an external magnetic field $\vec{H} = (0, 0, H_e)$ increasing at a constant rate $\frac{d\vec{H}}{dt} = \dot{H}$=const. According to the Maxwell equation (2), a vortex electric field $\vec{E} = (0, E_e, 0)$ is present. Here $H_e$ is the magnitude of the external magnetic field and $E_e$ is the magnitude of the back-ground electric field. In accordance with the concept of the critical state, the current density and the electric field must be parallel: $\vec{E} \parallel \vec{j}$;. The thermal and electromagnetic boundary conditions for the Eqs. (1)-(4) have the form

$$\begin{aligned} &\kappa \frac{dT}{dx}\bigg|_{x=0} + w_0[T(0) - T_0] = 0\,, \qquad && T(L) = T_0\,, \\ &\frac{dE}{dx}\bigg|_{x=0} = 0\,, && E(L) = 0\,, \end{aligned} \tag{5}$$

For the plane geometry and for the boundary conditions $H(0) = H_e, \quad H(L) = 0$, the magnetic field distribution is $H(x) = H_e(L - x)$, where $L = \frac{cH_e}{4\pi j_c}$ is the depth of magnetic

flux penetration into the sample and $w_0$ is the coefficient of heat transfer to the cooler at the equilibrium temperature $T_0$.

The condition of applicability of Eqs. (1)-(4) to the description of the dynamics of evolution of thermomagnetic perturbations are discussed at length in [1].

In the quasi-stationary approximation, terms with time derivatives can be neglected in Eqs. (1)-(4). This means that the heat transfer from the sample surface compensates the energy dissipation arising in the viscous flow of magnetic flux in the medium with an effective conductivity $\sigma_f$. In this approximation, the solution to Eq. (2) has the form

$$E = \frac{\dot{H}}{c}(L - x). \tag{6}$$

Upon substituting this expression into Eq. (1) we get an inhomogeneous equation for the temperature distribution $T(x, t)$,

$$\frac{d^2\Theta}{d\rho^2} - \rho\Theta = f(\rho). \tag{7}$$

Here we introduced the following dimensionless variables

$$f(\rho) = -[1 + r\omega\rho]\frac{j_0}{aT_0}, \qquad \Theta = \frac{T - T_0}{T_0}, \qquad \rho = \frac{L - x}{r},$$

and the dimensionless parameters $\omega = \dfrac{\sigma_f \dot{H}}{c j_0}$, and, $r = \left(\dfrac{c\kappa}{a\dot{H}L^2}\right)^{1/3}$, where $r$ characterizes the spatial scale of the temperature profile inhomogeneity in the sample. Solutions to Eq. (7) are Airy functions, which can be expressed through Bessel functions of the order 1/3 [6]

$$\Theta(\rho) = C_1\rho^{1/2}K_{1/3}\left(\frac{2}{3}\rho^{3/2}\right) + C_2\rho^{1/2}I_{1/3}\left(\frac{2}{3}\rho^{3/2}\right) + \Theta_0(\rho), \tag{8}$$

$$\Theta_0(\rho) = \rho^{1/2}K_{1/3}\left(\frac{2}{3}\rho^{3/2}\right)\int_0^{\rho}[1 + r\omega\rho_1]\rho_1^{3/2}I_{1/3}\left(\frac{2}{3}\rho_1^{3/2}\right)d\rho_1 -$$

$$\rho^{1/2}I_{1/3}\left(\frac{2}{3}\rho^{3/2}\right)\int_0^{\rho}[1 + r\omega\rho_1]\rho_1^{3/2}K_{1/3}\left(\frac{2}{3}\rho_1^{3/2}\right)d\rho_1,$$

where $C_1$ and $C_2$ are integration constants, which are determined by the boundary conditions to be

$$C_1 = 0, \quad C_2 = \frac{-w_0 L\Theta(0) + \kappa \left.\dfrac{d\Theta}{d\rho}\right|_{\rho = \frac{L}{r}}}{\left[w_0\left(\dfrac{L}{r}\right)^{1/2} I_{1/3}\left(\dfrac{2}{3}\rho^{-3/2}\right) - 2\dfrac{d}{d\rho}\left(\rho^{1/2}I_{1/3}\left(\dfrac{2}{3}\rho^{3/2}\right)\right)\right]\Bigg|_{\rho = \frac{L}{r}}}$$

From the Maxwell equation (2), the temperature inhomogeneity parameter can be expressed in the form

$$\alpha = \frac{r}{L} = \left[\frac{4\pi\nu j_0}{aH_e^2}\frac{H_e}{\dot{H}t_\kappa}\right]^{1/3} \tag{9}$$

It is evident that $\alpha \sim 1$ near the threshold for a flux jump, when $\dfrac{aH_e^2}{4\pi\nu j_0} \sim 1$, even under the quasi-stationary heating condition $\dfrac{\dot{H}t_\kappa}{H_e} << 1$; where $t_\kappa = \dfrac{\nu L^2}{\kappa}$ is the characteristic time of the heat conduction problem.

Let us estimate the maximum heating temperature $\Theta_m$ in the isothermal case $w = \dfrac{\kappa}{L} \geq 1$. The solution to Eq. (7) can be represented in the form

$$\Theta(x) = \Theta_m - \rho_0\frac{(x-x_m)^2}{2}, \tag{10}$$

near the point at which the temperature is a maximum, $x = x_m$ [7].

With solution (10) being approximated near the point $x_m = \dfrac{L}{2}$ with the help of the thermal boundary conditions, the coefficient $\rho_0$ can be easily determined to be $\left(\dfrac{8}{L^2}\right)\Theta_m$ and the temperature can be written as

$$\Theta(x) = \Theta_m\left[1 - \frac{4}{L^2}\left(x - \frac{L}{2}\right)^2\right], \tag{11}$$

Substituting this solution into Eq.(7), the superconductor maximum heating temperature due to magnetic flux jumps can be estimated as

$$\Theta_m = \frac{\left[j_0 + \dfrac{\sigma_f\dot{H}}{c}(L-x_m)\right]\dfrac{\dot{H}}{c\kappa T_0}(L-x_m)}{\dfrac{\gamma}{L^2} - \dfrac{a\dot{H}}{c\kappa}(L-x_m)}. \tag{12}$$

For a typical situation, when $\dfrac{\gamma}{L^2} << \dfrac{a\dot{H}}{c\kappa}(L-x_m)$ the estimation for $\Theta_m$ is

$$\Theta_m \approx \left[j_0 + \frac{\sigma_f\dot{H}}{c}(L-x_m)\right]\frac{\dot{H}L^2}{c\kappa T_0}(L-x_m). \tag{13}$$

Here, the parameter $\gamma \sim 1$ (for a parabolic temperature profile $\gamma \sim 8$). It is easy to verify that for typical values of $j_0 = 10^6 A/cm^2$, $\dot{H} = 10^4$ Gs/sek, and $L = 0,01$ cm the heating is sufficiently low: $\Theta_m << 1$. In the case of poor sample cooling, $w = 1 - 10 erg/(cm^2sK)$, the $\Theta_m$ is

$$\Theta_m = \frac{\dot{H}j_0L^2}{cw_0T_0} \approx 0,5;$$

i.e., the heating temperature can be as high as $\delta T_m = T_0\Theta_m \sim 2K$. One can see that in the case of poor sample cooling, the heating can be rather noticeable and influences the conditions of the thermomagnetic instability of the critical state in the superconductor.

Let us investigate the stability of the critical state with respect to small thermal $\delta T$ and electromagnetic $\delta E$ fluctuations in the quasi-stationary approximation. We represent solutions to Eqs. (1)-(4) in the form

$$T(x,t) = T(x) + \exp\left\{\frac{\lambda t}{t_\kappa}\right\}\delta T\left(\frac{x}{L}\right),$$
$$E(x,t) = E(x) + \exp\left\{\frac{\lambda t}{t_\kappa}\right\}\delta E\left(\frac{x}{l}\right), \tag{14}$$

where $T(x)$ and $E(x)$ are solutions to the unperturbed equations obtained in the quasi-stationary approximation describing the background distributions of temperature and electric field in the sample and $\lambda$ is a parameter to be determined. The instability region is determined by the condition that $Re\lambda \geq 0$. From solution (14), one can see that the characteristic time of thermal and electromagnetic perturbations $t_j$ is of the order of $t_\kappa/\lambda$. Linearizing Eqs. (1)-(4) for small perturbations $\left(\frac{\delta T}{T(x)}, \frac{\delta E}{E(x)} << 1\right)$ we obtain the following equations in the quasi-stationary approximation:

$$\nu\frac{\lambda}{t_\kappa}\delta T = \frac{\kappa}{L^2}\frac{d^2\delta T}{dx^2} + \left[j(x) + \sigma_f E(x)\right]\delta E - aE(x)\delta T,$$
$$\frac{1}{L^2}\frac{d^2\delta E}{dx^2} = \frac{4\pi\lambda}{c^2 t_\kappa}\left[\sigma_f\delta E - a\delta T\right]. \tag{15}$$

Eliminating the variable $\delta T$ between Eqs. (15), we obtain a fourth- order differential equation with variable coefficients for the electromagnetic field $\delta E$:

$$\frac{d^4\delta E}{dz^4} - \left[\lambda(1+\tau) + \frac{E(z)}{E_\kappa}\right]\frac{d^2\delta E}{dz^2} + \lambda\left[\lambda\tau - B(z)\right]\delta E = 0. \tag{16}$$

Here, we introduced the following dimensionless variables: $z = \frac{x}{L}$, $B(z) = \frac{4\pi a L^2}{c^2\nu}j(z)$, $j(z) = \sigma_f E(z) - a[T(z) - T_0]$, $E(z) = \frac{\dot{H}L}{c}(1-z)$, $\tau = \frac{4\pi\sigma_f\kappa}{c^2\nu}$, $E_\kappa = \frac{\kappa}{aL^2}$. $\nu = \nu_0\left(\frac{T}{T_0}\right)^3$, $\kappa = \kappa_0\left(\frac{T}{T_0}\right)$.

One should keep in mind that the variable $T(z)$ and $E(z)$ are given by Eq. (8), in which $\rho = \frac{L}{r}(1-z)$. Using the relation between $\delta T$ and $\delta E$ given by Eqs. (15), we write the boundary conditions to Eq. (16) in the form:

$$\left.\frac{d^2\delta E}{dz^2}\right|_{z=1} = 0, \quad \left.\frac{d^3\delta E}{dz^3}\right|_{z=0} = -W\left.\left[\frac{d^2\delta E}{dz^2} - \lambda\tau\delta E\right]\right|_{z=0},$$
$$\delta E|_{z=1} = 0, \quad \left.\frac{d\delta E}{dz}\right|_{z=0} = 0. \tag{17}$$

where $W = \dfrac{w_0 L}{\kappa}$ is the dimensionless thermal impedance.

The condition for the existence of a nontrivial solution to Eq. (16) subject to boundary conditions (17) allows one to determine the boundary of the critical-state thermomagnetic instability in a superconducting sample. This problem is complicated, and its analytical solution cannot be found in a closed form. We will consider the development of thermomagnetic instability in the dynamical approximation, which is valid for composite superconductors with high value of $\sigma_f E$.

The dynamical character of the instability development leads to the predominance of heat diffusion over magnetic flux diffusion in the sample: $\tau = \dfrac{D_t}{D_m} >> 1$ [1], where $D_t = \dfrac{\kappa}{\nu}$ and $D_m = \dfrac{c^2}{4\pi\sigma_f}$ are the coefficients of the thermal and magnetic diffusion, respectively.

In this case, as seen from Eq. (14), the characteristic times $t_j$ of temperature and electromagnetic field perturbations have to satisfy the inequalities $t_j >> t_\kappa$ ($\lambda << 1$) and $t_j << t_m$ ($\lambda\tau >> 1$), where $t_\kappa = \dfrac{L^2}{D_t}$ and $t_m = \dfrac{L^2}{D_m}$ are the characteristic times of the thermal and magnetic diffusion, respectively.

As well known that ([3, 4]), the effective magnitude of conductivity $\sigma_f$ is greater in composite superconductors than the one in hard superconductors. We can assume that induced normal current $\sigma_f E$ compensates the decreasing of critical current $j_c(T)$, caused by increasing of temperature and obviously prevents magnetic flow penetration into the sample. In this case we can neglect the moving of magnetic flow. In other words, in composites the development of thermomagnetic instability is accompanied with "slow" vortices, with characteristic time of increasing $t_j \sim \dfrac{t_\kappa}{\lambda} >> t_\kappa$ ($\lambda << 1$) or $t_m >> \dfrac{t_m}{\lambda\tau} = \dfrac{t_\kappa}{\lambda} = t_j$ ($\lambda\tau >> 1$).

In the approximation ($\tau >> 1$, $\lambda\tau >> 1$, $\lambda << 1$) Eq. (16) is reduced to a lower order differential equation

$$\frac{d^2\delta E}{dz^2} + \left[\lambda - \frac{B(z)}{\tau}\right]\delta E = 0\,. \tag{18}$$

In the case of $\tau >> 1$, the instability threshold depends on the electrodynamic boundary conditions at the surface of the sample only slightly. Therefore, the thermal boundary conditions at the boundaries of the current-carrying layer ($z = 0, z = 1$) can be neglected and one can keep only the electrodynamic boundary conditions to Eq. (18), bacause under the condition of adiabatic instability development in the composite superconductor, the instability threshold depends on the conditions of heat removal on the sample surface only weakly [1].

Multiplying Eq. (18) by $\delta E$ and integrating the result with respect to $z$ over the interval $0 < z < 1$, we obtain

$$\lambda = \frac{\frac{1}{\tau}\int\limits_0^1 B(z)\delta E^2 dz - \int\limits_0^1 \left(\frac{d\delta E}{dz}\right)^2 dz}{\int\limits_0^1 \delta E^2 dz} \tag{19}$$

where we use the equality

$$\int\limits_0^1 \frac{d^2\delta E}{dz^2}\delta E dz = \delta E(z)\left(\frac{d\delta E}{dz}\right) - \int\limits_0^1 \left(\frac{\delta E}{dz}\right)^2 dz = -\int\limits_0^1 \left(\frac{\delta E}{dz}\right)^2 dz$$

and the boundary conditions. The right-hand side of Eq. (19) has a minimum at $\lambda = \lambda_c$:

$$\lambda_c = \frac{\frac{1}{\tau}\int\limits_0^1 B(z)\delta E^2 dz}{\int\limits_0^1 \delta E^2 dz}. \tag{20}$$

Then

$$\frac{1}{\tau}\int\limits_0^1 B(z) n_E^2 dz = \frac{\int\limits_0^1 \left(\frac{d\delta E}{dz}\right)^2 dz}{\int\limits_0^1 \delta E^2 dz} \tag{21}$$

where

$$n_E^2 = \frac{\delta E^2}{\int\limits_0^1 \delta E^2 dz}, \qquad \int\limits_0^1 n_E^2 dz = 1.$$

Since we do not know the function $\delta E(z)$, we try, following [8], to obtain an integral estimation of the instability growth increment and the low boundary of its occurrence. The behavior of the integrand in Eq. (21) is basically determined by the factor $E = \frac{\dot{H}L}{c}(1-z)$, which is equal to zero at $z = 1$ (the other factors change more smoothly). Hence, the integrand reaches its maximum at $z = 0$ and the upper estimate for $\lambda_c$ is

$$\lambda_c \leq \frac{B(0)}{\tau}. \tag{22}$$

It is evident that $\lambda_c >> 1$ and $\lambda_c \tau << 1$ at $\tau << 1$. Numerical evaluation gives $\lambda_c \approx 10^{-1} \div 10^{-3}$ at $\tau = 10^3$.

Equations (21) and (22) enable one to write the instability occurrence criterion in the form

$$\frac{1}{\tau}\int_0^1 B(z)n_E^2 dz > \frac{\int_0^1 \left(\frac{d\delta E}{dz}\right)^2}{\int_0^1 \delta E^2 dz}\,. \tag{23}$$

This criterion essentially depends on the boundary conditions and the functions $j(z)$, $E(z)$, $T(z)$ (see, Ref.[7]). Inequality (23) can be strengthened by means of an evaluation,

$$\int_0^1 \left(\frac{d\delta E}{dz}\right)^2 dz > \frac{\pi^2}{4}\int_0^1 \delta E^2 dz \tag{24}$$

which can be easily verified by expanding the function $\delta E(z)$ in a Fourier series:

$$\delta E(z) = A_m cos\frac{\pi z(2m+1)}{2}\,.$$

Let us now try to strengthen inequality (24) further. For this purpose, we consider the integral

$$\int_0^1 g(z)(n_E^2-1)dz = \int_0^{z_1} g(z)(n_E^2-1)dz + \int_{z_1}^0 g(z)(n_E^2-1)dz.$$

The last term can be represented in the form

$$\int_0^1 g(z)(n_E^2-1)dz = (g_+ - g_-)\int_0^1 g(z)(n_E^2-1)dz,$$

taking intermediate values of the $g_-$ in the range $z < z_1$ and $g > g_-$ in the range $z_1 < z < 1$ outside the integral. It is evident that [7]

$$\int_0^1 g(z)(n_E^2-1)dz \leq 0$$

or

$$\int_0^1 g(z)n_E^2 dz \leq \int_0^1 g(z)dz. \tag{25}$$

With inequality (25), the instability occurrence criterion can be represented in the form

$$\int_0^1 B(z)dz \geq \frac{\pi^2}{4}\tau\,. \tag{26}$$

Inequality (26), unlike the analogous criterion for a homogeneous temperature profile, has an integral character and takes into account the influence of each part of the superconductor on the threshold for the superconducting-state instability. If condition (26) is satisfied,

then small fluctuations of temperature $\delta T$ and electric field $\delta E$ in the superconductor will exponentially increase with time. The most probable result of the development of such an instability would be a transition from a critical state to a resistive one.

In conclusion, we note that the description of the critical state applied in this paper is based on the BCS microscopic theory. Then, a continual approximation is used and the physical parameters are assumed to vary slowly at distances of the order of the average distance $d$ between vortices. The continuity condition is, therefore, $L >> d$.

One more limitation occurs from possible superconductor overheating above the critical temperature $T_c$, where Eqs. (1)-(4) are not valid. This case is realized if the condition $j_c(T_0, E) \geq a(T_c - T_0)$ is fulfilled.

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**NIZAM A.TAYLANOV**

**Address:**
Theoretical Physics Department and
Institute of Applied Physics,
National University of Uzbekistan,
Vuzgorodok, 700174, Tashkent, Uzbekistan
Telephone:(9-98712),461-573, 460-867.
fax: (9-98712) 463-262,(9-98712) 461-540,(9-9871) 144-77-28
e-mail: taylanov@iaph.tkt.uz